PERGAMON

International Journal of Solids and Structures 37 (2000) 1065–1078

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

A further investigation of Green's functions for a piezoelectric material with a cavity or a crack

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Received 1 July 1998; in revised form 9 April 1999

Abstract

In this paper, the Green's functions of an infinite two-dimensional piezoelectric material containing an elliptical cavity are re-investigated by introducing exact electric boundary conditions on the hole boundary, and the corresponding modified solutions are obtained. By setting ϵ_0, the dielectric constant in the cavity, to be zero, the modified Green's functions are returned to the conventional ones (Lu and Williams, 1998). Furthermore, the hoop stresses along the hole boundary under the action of a set of generalized concentrated forces are obtained. When the hole is reduced to a slit crack, the expressions of generalized stress intensity factors are also provided. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Piezoelectric material; Green's function; Inclusion; Stroh formalism

1. Introduction

The problems of Green's functions for two or three-dimensional piezoelectric materials have been studied by many researchers (e.g. Dunn, 1994a; Norris, 1994; Khutoryansky and Sosa, 1995; Lu and Williams, 1998). The electroelastic Green's functions play an important role in the analysis of piezoelectric inclusion and inhomogeneity problems, defect and crack problems, as well as stress and electric concentration problems. Especially, the Green's functions can be used as fundamental solutions of well used boundary element method for solving piezoelectric problems with finite boundary sizes and subjected to general mechanical–electric loading (Lu and Mahrenholtz, 1994; Lee and Jiang, 1994;

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PII: S0020-7683(99)00117-1

Khutoryansky and Sosa, 1995; Liang and Hwu, 1996). Therefore, an in-depth investigation and understanding of the properties for the mechanical–electric coupled Green's functions is essential.

Due to the similarities of the expressions between elasticity and piezoelectricity, the solutions for elastic problems usually can be extended to relevant piezoelectric problems following proper regularities in many cases. However, care should be taken for some situations because the direct extensions may lead to physically incorrect results. One of the examples is given by considering a piezoelectric material containing a cavity or a crack with free surface. The solutions for this kind of the problems were obtained by extending the relevant results of elastic medium with traction free holes (for examples, Sosa, 1991; Pak, 1992; Chung and Ting, 1996; Lu and Williams, 1998). In physical terms, this kind of extensions implies that the boundary condition along the cavity or crack surface is assumed to be traction free and electric impermeable, i.e. ignoring the electric field within the cavity or crack. Recent studies have shown that this commonly used electric boundary condition in the literature could lead to physically unreasonable results, especially in the case of cracks (Dunn, 1994b; Sosa and Khutoryansky, 1996; Zhang et al., 1998). Therefore, the current work in the problems has tended to use the exact electric boundary condition, i.e. the normal component of electric displacement is continuous across the cavity or crack surface. In view of this fact, some of the previous results based on the simplified electric conditions should be improved.

In this paper, the Green's functions of piezoelectric material with an elliptical cavity are re-investigated. The solutions were obtained by Lu and Williams (1998) based on the assumption of electric impermeability on the surface of the cavity. As indicated above, the solutions may result in considerable error when the cavity becomes a very slender ellipse or a sharp crack. Due to the important analytical and numerical applications of the Green's functions in solving piezoelectric materials with defects, the formulations obtained previously should be modified by introducing the correct electrical boundary conditions.

Recently, Ting (1996) investigated Green's functions for an infinite anisotropic elastic medium with an elliptical inclusion of dissimilar material. By carefully including image singularities in the solutions, the constructed Green's functions subjected to a singularity outside, inside, or on the interface of the elliptical inclusion can be reduced to the same solution when the applied singularity is on the elliptical boundary. Furthermore, the paper provides a regular way to construct the Green's functions of the subject. In this paper, the suggested method by Ting (1996) is extended to determine the Green's functions for a piezoelectric medium with an elliptical cavity.

2. Basic equations

In a fixed rectangular coordinate system x_i ($i = 1, 2, 3$), the linear constitutive relations for piezoelectric materials are given by

$$\sigma_{ij} = C_{ijkm}\gamma_{km} - e_{mij}E_m,\ D_i = e_{ikm}\gamma_{km} + \epsilon_{im}E_m, \tag{1}$$

where σ_{ij}, γ_{ij}, D_i and E_k are stress, strain, electric displacement and electric field components, respectively; C_{ijkm}, e_{mij} and ϵ_{im} are elastic, piezoelectric and dielectric constants, respectively, which satisfy the following symmetries:

$$C_{ijkm} = C_{jikm} = C_{ijmk} = C_{kmij},\ e_{mij} = e_{mji},\ \epsilon_{im} = \epsilon_{mi}. \tag{2}$$

If u_i are mechanical displacements and ϕ the electric potential, the deformation relations are

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),\ E_i = -\phi_{,i}. \tag{3}$$

In the absence of body forces and free charges, the governing equations of linear piezoelectricity are

$$\sigma_{ij,i} = 0,\ D_{i,i} = 0. \tag{4}$$

To treat the elastic and electric variables on an equal footing and reduce the amount of writing, the notation introduced by Barnett and Lothe (1975) is reviewed here. Defining

$$U_K = \begin{cases} u_k, & K = 1,2,3, \\ \phi, & K = 4, \end{cases} \tag{5}$$

$$Z_{Km} = \begin{cases} \gamma_{km}, & K,m = 1,2,3, \\ E_m, & K = 4;\ m = 1,2,3, \end{cases} \tag{6}$$

$$\Sigma_{iJ} = \begin{cases} \sigma_{ij}, & i,J = 1,2,3, \\ D_i, & i = 1,2,3;\ J = 4, \end{cases} \tag{7}$$

$$E_{iJKm} = \begin{cases} C_{ijkm}, & i,J,K,m = 1,2,3, \\ e_{mij}, & i,J,m = 1,2,3;\ K = 4, \\ e_{ikm}, & i,K,m = 1,2,3;\ J = 4, \\ -\epsilon_{im}, & i,m = 1,2,3;\ J,K = 4, \end{cases} \tag{8}$$

in which lower case subscripts take on the range 1–3, while the capital subscripts take on the range 1–4, and repeated capital subscripts are summed over 1–4. With this notation, the constitutive equations Eq. (1), and the governing equations Eq. (4) can be expressed in shorthand form as

$$\Sigma_{iJ} = E_{iJKm} Z_{Km} = E_{iJKm} U_{K,m} \tag{9}$$

$$\Sigma_{iJ,i} = 0. \tag{10}$$

It is seen that the governing equations for piezoelectric problems have similar expressions to those for elastic ones. Therefore, some solutions for elastic problems can be extended to relevant piezoelectric ones with proper modifications.

For two-dimensional deformations in which u_k and ϕ, or U_K, depend on x_1 and x_2 only, an extended version of Stroh formalism satisfying Eqs. (9) and (10) has been developed (see, for example, Chung and Ting, 1996; Lu and Williams, 1998). It is summarized here. The general solution to Eq. (10) could be expressed as

$$\mathbf{u} = \{u_k, \phi\}^T = \alpha f(z),\ z = x_1 + p x_2, \tag{11}$$

in which $f(z)$ is an arbitrary function of z and p and $\mathbf{a}$ satisfy the relation

$$\left\{\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2 \mathbf{T}\right\}\mathbf{a} = \mathbf{0}, \tag{12}$$

where $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ are 4×4 matrices:

$$Q_{IK} = E_{1IK1},\ R_{IK} = E_{1IK2},\ T_{IK} = E_{2IK2}. \tag{13}$$

Let the generalized stress functions, ψ, be defined as

$$\psi = \mathbf{b}f(z), \tag{14}$$

where

$$\mathbf{b} = (\mathbf{R}^T + p\mathbf{T})\mathbf{a} = -\frac{1}{p}(\mathbf{Q} + p\mathbf{R})\mathbf{a}. \tag{15}$$

The elastic stress and the electric displacements can then be expressed as

$$\begin{Bmatrix} \sigma_{2j} \\ D_2 \end{Bmatrix} = \psi_{,1}, \begin{Bmatrix} \sigma_{1j} \\ D_1 \end{Bmatrix} = -\psi_{,2}. \tag{16}$$

It is known that the four order equations Eq. (12) are an eigenvalue problem, which gives four pairs of complex conjugates and corresponding vectors. Let p_α ($\alpha = 1, 2, 3, 4$) be eigenvalues with $\mathrm{Im}\{p_\alpha\} > 0$, $\mathbf{a}_\alpha$ and $\mathbf{b}_\alpha$ the associated eigenvectors, the general solutions for the generalized displacements $\mathbf{u}$ and generalized stress functions ψ can thus be written as (Ting, 1996)

$$\mathbf{u} = \frac{1}{\pi}\,\mathrm{Im}\{\mathbf{A}\langle f(z_*)\rangle\mathbf{q}\},\ \psi = \frac{1}{\pi}\,Im\{\mathbf{B}\langle f(z_*)\rangle\mathbf{q}\}, \tag{17}$$

where

$$\langle f(z_*)\rangle = \mathrm{diag}[f(z_1), f(z_2), f(z_3), f(z_4)],\ \mathbf{q} = \{q_1, q_2, q_3, q_4\}^T,$$

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4],\ \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4], \tag{18}$$

and $z_\alpha = x_1 + p_\alpha x_2$. The arbitrary functions $f(z)$ and the unknown constant vector $\mathbf{q}$ in Eq. (17) are determined by the boundary conditions set for each particular problem.

It is seen that the extended Stroh formalism for piezoelectricity has the same form as the Stroh formalism for anisotropic electricity. Therefore, most properties and identities existing in the Stroh formalism for anisotropic elasticity can be extended to piezoelectric problems.

3. Boundary conditions of a piezoelectric material with an elliptical cavity

Consider an infinite two-dimensional anisotropic piezoelectric material containing an elliptical cavity, described by Ω and Ω_c, respectively, and subjected to a concentrated mechanical force and electric charge density vector $\mathbf{p}$ at $\mathbf{x}^0 = (x_1^0, x_2^0)$ outside the ellipse, as shown in Fig. 1. The contour Γ is given by

$$x_1 = a\cos\vartheta,\ x_2 = b\sin\vartheta, \tag{19}$$

where ϑ is a real parameter, and a and b are the semi-major and semi-minor axes of the ellipse, respectively. The unit tangential vector $\mathbf{n}$ and the unit normal vector $\mathbf{m}$ are given by

$$\mathbf{n} = \{\cos\theta, \sin\theta, 0\}^T,\ \mathbf{m} = \{-\sin\theta, \cos\theta, 0\}^T, \tag{20}$$

where θ is directed counterclockwise from the positive x_1-axis to the direction of $\mathbf{n}$. From Eq. (16), the traction and electric displacement components in the piezoelectric material medium, along the hole surface, can be expressed as

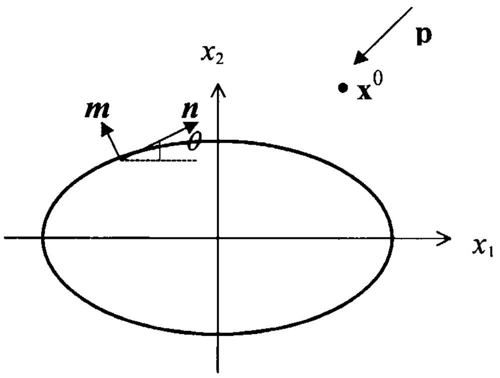

Fig. 1. An infinite two-dimensional piezoelectric material with an elliptic cavity under a concentrated force vector **p** acting at point $\mathbf{x}^0$.

$$\mathbf{t}_m = \{\mathbf{s}\cdot\mathbf{m},\ \mathbf{\Delta}\cdot\mathbf{m}\}^T = \{t_1,\ t_2,\ t_3,\ D_m\}^T = \psi_{,n}, \tag{21}$$

in which t_j is the component of surface traction vector.

The cavity is assumed to be filled with a homogeneous gas of dielectric constant ϵ_0, and is free of forces and surface charge density (Sosa and Khutoryansky, 1996). Therefore, the induced electric displacements $D_i^c$and electric fields E_i^c exist in Ω_c and can be described via the constitutive relation

$$D_i^c = \epsilon_0 E_i^c \tag{22}$$

and the deformation relation

$$E_i^c = -\phi_{,i}^c, \tag{23a}$$

respectively. The electric potential ϕ^c in Ω_c can be solved from an equilibrium equation, i.e.

$$D_{i,i}^c = \phi_{,ii}^c = 0. \tag{23b}$$

Following the treatment of the previous section, the general solution to Eq. (23b) can be expressed as

$$\phi^c = f^c(z),\ z = x_1 + ix_2, \tag{24}$$

where $f^c(z)$ is an arbitrary function of z. It can be verified that Eq. (24) has satisfied Eq. (23b). Similarly, defining

$$\psi^c = b^c f^c(z),\ b^c = -i\epsilon_0, \tag{25}$$

the electric displacements D_i^c, can therefore be written as

$$D_1^c = -\psi_{,2}^c,\ D_2^c = \psi_{,1}^c. \tag{26}$$

In this way, the normal component of the electric displacement can be expressed as

$$D_m^c = \mathbf{D}^c\cdot\mathbf{m} = \psi_{,n}^c. \tag{27}$$

The boundary conditions along the surface of the cavity is prescribed such that the boundary is traction free and the normal component of the electric displacement as well as the electric potential are continuous across the surface, i.e.

$$t_j = 0,\ D_m = D_m^c,\ \phi = \phi^c. \tag{28}$$

In the previous work by Lu and Williams (1998), Eq. (28b) and (28c) were approximated by the single equation $D_m = 0$ (the so-called condition of impermeability). By using Eqs. (21) and (27), after proper treatment, Eq. (28) can be equivalently expressed as

$$\mathbf{e}^T\mathbf{u} = \phi^c,\ \psi = \mathbf{e}\psi^c,\ \mathbf{e} = [0,\ 0,\ 0,\ 1]^T, \tag{29}$$

where $\mathbf{u}$ and ψ are four order vectors given by Eq. (17), ϕ^c and ψ^c are the scalar functions in the cavity. Eq. (29) provides a concise form of the general boundary conditions defined on the cavity.

4. Green's functions

4.1. Conformal mapping

It is known that the transformation functions,

$$z_\alpha = c_\alpha\varsigma_\alpha + d_\alpha\varsigma_\alpha^{-1}, \tag{30}$$

can map the region Ω outside the ellipse in the z_α-plane onto the outside of a unit circle in the ς-plane. In Eq. (30), $c_\alpha = 1/2(a - ip_\alpha b)$ and $d_\alpha = 1/2(a - ip_\alpha b)$. Since the roots of $dz_\alpha/d\varsigma_\alpha$ are located inside the unit circle, the transformations Eq. (30) are one-to-one outside the hole and $\varsigma_\alpha|_\Gamma = e^{i\vartheta}$ when z_α are on the elliptical boundary, in which ϑ is defined in Eq. (19). On the other hand, the mapping of the region Ω_c inside the ellipse is done by considering a straight line Γ_0 along x_1 and of length $2\sqrt{a^2 - b^2}$ (Sosa and Khutoryansky, 1996). The function,

$$z = \frac{1}{2}(a + b)(\varsigma + t\varsigma^{-1}), \tag{31}$$

can therefore map the region in the z-plane enclosed by the ellipse excluding the line Γ_0 onto the region in the ς-plane of an annular ring between the unit circle $\varsigma|_\Gamma = e^{i\vartheta}$ and the inner circle $\varsigma|_{\Gamma_0} = \sqrt{t}e^{i\vartheta}$, in which $t = (a - b)/(a + b)$. In the annular ring, a holomorphic function $\Phi(\varsigma)$ can be expressed by Laurent's expansion as

$$\Phi(\varsigma) = \sum_{k=1}^{\infty} a_{-k}\varsigma^{-k} + \sum_{k=0}^{\infty} a_k\varsigma^k \quad \sqrt{t} \le |\varsigma| \le 1. \tag{32}$$

To ensure the function is single valued inside the cavity and along the line Γ_0, the following condition must be satisfied (Sosa and Khutoryansky, 1996):

$$\Phi(\sqrt{t}e^{i\vartheta}) = \Phi(\sqrt{t}e^{-i\vartheta}), \tag{33}$$

from which the relations between the coefficients a_{-k} and a_k in Eq. (32) can be determined as $a_{-k} = t^k a_k$. Therefore, Eq. (32) can be further written as

$$\Phi(\varsigma) = \sum_{k=1}^{\infty} a_k f_k(\varsigma), \tag{34}$$

where

$$f_k(\varsigma) = \varsigma^k + t^k \varsigma^{-k} \tag{35}$$

is continuous inside the ellipse. Eq. (35) can be used to construct solutions within the ellipse.

4.2. Construction of Green's functions

The Green's functions for a piezoelectric material with an elliptical hole subject to electric impermeable condition was studied by Lu and Williams (1998). Although not mentioned there, the terms concerning image singularities were included in the solutions. Ting (1996) thoroughly discussed the necessity of including image singularities in constructing a Green's function. With knowledge of this, and making use of the results of Eqs. (17), (24) and (25), the general solutions in the piezoelectric material and in the cavity for the present problem can be written as (Ting, 1996; Lu and Williams, 1998)

$$\mathbf{u} = \frac{1}{\pi}\,\mathrm{Im}\left\{\mathbf{A}\langle \ln(\varsigma_* - \varsigma_{*0})\rangle \mathbf{q}\right\} + \frac{1}{\pi}\,\mathrm{Im}\sum_{\beta=1}^{4}\left\{\mathbf{A}\langle \ln(\varsigma_*^{-1} - \bar{\varsigma}_{\beta 0})\rangle \mathbf{q}_\beta\right\} + \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{\infty}\frac{1}{k}\left\{\mathbf{A}\langle \varsigma_*^{-k}\rangle \mathbf{g}_k\right\},$$

$$\psi = \frac{1}{\pi}\,\mathrm{Im}\left\{\mathbf{B}\langle \ln(\varsigma_* - \varsigma_{*0})\rangle \mathbf{q}\right\} + \frac{1}{\pi}\,\mathrm{Im}\sum_{\beta=1}^{4}\left\{\mathbf{B}\langle \ln(\varsigma_*^{-1} - \bar{\varsigma}_{\beta 0})\rangle \mathbf{q}_\beta\right\} + \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{\infty}\frac{1}{k}\left\{\mathbf{B}\langle \varsigma_*^{-k}\rangle \mathbf{g}_k\right\} \tag{36}$$

and

$$\phi^{c} - \phi_0^{c} = \frac{1}{\pi}\,\mathrm{Im}\sum_{\beta=1}^{4}\left\{\ln\left(z - \hat{z}_\beta\right) q_\beta^{c}\right\} + \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{\infty}\frac{1}{k}\left\{f_k(\varsigma) h_k^{c}\right\},$$

$$\psi^{c} - \psi_0^{c} = \frac{1}{\pi}\,\mathrm{Im}\sum_{\beta=1}^{4}\left\{b^{c}\ln\left(z - \hat{z}_\beta\right) q_\beta^{c}\right\} + \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{\infty}\frac{1}{k}\left\{b^{c} f_k(\varsigma) h_k^{c}\right\}. \tag{37}$$

In Eqs. (36) and (37), the unknowns $\mathbf{q}_\beta$, $\mathbf{g}_k$, q_β^{c}, h_k^{c}, ϕ_0^{c} and ψ_0^{c} are constants to be determined, $f_k(\varsigma)$ is given by Eq. (35), and

$$\mathbf{q} = \mathbf{A}^T \mathbf{p},$$

$$x_1^0 + p_\alpha x_2^0 = z_{\alpha 0} = c_\alpha \varsigma_{\alpha 0} + d_\alpha \varsigma_{\alpha 0}^{-1},\ \hat{x}_1^\beta + i\hat{x}_2^\beta = \hat{z}_\beta = \frac{1}{2}(a+b)\left(\varsigma_{\beta 0} + t\varsigma_{\beta 0}^{-1}\right), \tag{38}$$

from which $\varsigma_{\alpha 0}$ and $\hat{z}_\beta$ can be defined. $\langle\cdot\rangle$ indicates a four-order diagonal matrix. The meaning of each term in the solutions of Eqs. (36) and (37) has been illustrated in detail by Ting (1996). It should be noticed that the solutions within the cavity are scalar functions. Now, the problem is reduced to determine the unknowns given above, according to the boundary conditions (29).

4.3. Determination of unknown constants

On the elliptical boundary, Eq. (36a) is reduced to

$$\mathbf{u}(\sigma)=\frac{1}{\pi}\operatorname{Im}\left\{-\bar{\mathbf{A}}\langle\ln(\bar{\sigma}-\bar{\varsigma}_{*0})\rangle\bar{\mathbf{q}}+\sum_{\beta=1}^{4}\ln\left(\sigma^{-1}-\bar{\varsigma}_{\beta 0}\right)\mathbf{A}\mathbf{q}_\beta+\sum_{k=1}^{\infty}\frac{1}{k}\sigma^{-k}\mathbf{A}\mathbf{g}_k\right\}. \tag{39}$$

In the above expression, the relation $\operatorname{Im}(F)=-\operatorname{Im}(\bar{F})$ has been used. By writing

$$z-\hat{z}_\beta=\frac{1}{2}(a+b)(\varsigma-\varsigma_{\beta 0})\left(1-\hat{\tau}_\beta\varsigma^{-1}\right),\ \hat{\tau}_\beta=t\varsigma_{\beta 0}^{-1}, \tag{40}$$

Eq. (37a) on the elliptical boundary, can be expressed as

$$\phi^{c}(\sigma)-\phi_0^{c}=\frac{1}{\pi}\operatorname{Im}\left\{\sum_{\beta=1}^{4}\left\{\ln\left[\frac{1}{2}(a+b)\right]q_\beta^{c}-\ln\left(\sigma^{-1}-\bar{\varsigma}_{\beta 0}\right)\bar{q}_\beta^{c}+\ln\left(1-\hat{\tau}_\beta\sigma^{-1}\right)q_\beta^{c}\right\}+\sum_{k=1}^{\infty}\frac{1}{k}\sigma^{-k}\left\{-\bar{h}_k+t^k h_k\right\}\right\}. \tag{41}$$

The expressions of ψ and ψ^{c} on the elliptical boundary can be similarly obtained. Substituting these relations into the boundary conditions (29), and with the use of the series representation,

$$\ln(1-x)=-\sum_{k=1}^{\infty}\frac{1}{k}x^k, \tag{42}$$

one has

$$\phi_0^{c}=-\frac{1}{\pi}\operatorname{Im}\left\{\ln\left[\frac{1}{2}(a+b)\right]\sum_{\beta=1}^{4}q_\beta^{c}\right\},\ \psi_0^{c}=-\frac{1}{\pi}\operatorname{Im}\left\{b^{c}\ln\left[\frac{1}{2}(a+b)\right]\sum_{\beta=1}^{4}q_\beta^{c}\right\}, \tag{43}$$

$$\mathbf{e}^T\left\{-\bar{\mathbf{A}}\mathbf{I}_\beta\bar{\mathbf{q}}+\mathbf{A}\mathbf{q}_\beta\right\}=-\bar{q}_\beta^{c},\ -\bar{\mathbf{B}}\mathbf{I}_\beta\bar{\mathbf{q}}+\mathbf{B}\mathbf{q}_\beta=-\mathbf{e}\bar{b}^{c}\bar{q}_\beta^{c}, \tag{44}$$

$$\mathbf{e}^T\mathbf{A}\mathbf{g}_k+\bar{h}_k^{c}=t^k h_k^{c}-\sum_{\beta=1}^{4}\hat{\tau}_\beta^k q_\beta^{c},\ \mathbf{B}\mathbf{g}_k+\mathbf{e}\bar{b}^{c}\bar{h}_k^{c}=\mathbf{e}b^{c}\left\{t^k h_k^{c}-\sum_{\beta=1}^{4}\hat{\tau}_\beta^k q_\beta^{c}\right\}. \tag{45}$$

In the above

$$\mathbf{I}_1=\operatorname{diag}[1,\ 0,\ 0,\ 0],\ \mathbf{I}_2=\operatorname{diag}[0,\ 1,\ 0,\ 0],\ \mathbf{I}_3=\operatorname{diag}[0,\ 0,\ 1,\ 0],\ \mathbf{I}_4=\operatorname{diag}[0,\ 0,\ 0,\ 1]. \tag{46}$$

From Eq. (44), $\mathbf{q}_\beta$ and q_β^{c} can be determined:

$$\mathbf{q}_\beta=\mathbf{N}_1^{-1}\bar{\mathbf{N}}_2\mathbf{I}_\beta\bar{\mathbf{q}},\ q_\beta^{c}=\mathbf{e}^T\left\{\mathbf{A}-\bar{\mathbf{A}}\bar{\mathbf{N}}_1^{-1}\mathbf{N}_2\right\}\mathbf{I}_\beta\mathbf{q}, \tag{47}$$

where $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{H}_\epsilon$ are defined as

$$\mathbf{H}_\epsilon=b^{c}\mathbf{I}_4\mathbf{A}=-i\epsilon_0\mathbf{I}_4\mathbf{A},\ \mathbf{N}_1=\mathbf{B}+\mathbf{H}_\epsilon,\ \mathbf{N}_2=\mathbf{B}-\mathbf{H}_\epsilon. \tag{48}$$

Constants $\mathbf{g}_k$ and h_k^{c} can be determined as follows. From Eq. (45), we have

$$\bar{h}_k^{\mathrm{c}}\mathbf{e} = -\frac{1}{2\bar{b}^{\mathrm{c}}}(\mathbf{B} - \mathbf{H}_\epsilon)\mathbf{g}_k. \tag{49}$$

Substitution of Eq. (49) into Eq. (45b) leads to

$$(\mathbf{B} + \mathbf{H}_\epsilon)\mathbf{g}_k = -t^k\big(\bar{\mathbf{B}} - \bar{\mathbf{H}}_\epsilon\big)\bar{\mathbf{g}}_k - 2b^{\mathrm{c}}y_k\mathbf{e}, \tag{50}$$

where

$$y_k = \sum_{\beta=1}^{4}\hat{\tau}_\beta^k q_\beta^{\mathrm{c}} = \sum_{\beta=1}^{4} t^k \varsigma_{\beta 0}^{-k} q_\beta^{\mathrm{c}}. \tag{51}$$

Therefore, $\mathbf{g}_k$ can be obtained by solving Eq. (50), i.e.

$$\mathbf{g}_k = -2b^{\mathrm{c}}\hat{\mathbf{g}}_k, \tag{52}$$

where

$$\hat{\mathbf{g}}_k = \Big\{\mathbf{N}_1 - t^{2k}\bar{\mathbf{N}}_2\bar{\mathbf{N}}_1^{-1}\mathbf{N}_2\Big\}^{-1}\Big\{y_k + t^k\bar{y}_k\bar{\mathbf{N}}_2\bar{\mathbf{N}}_1^{-1}\Big\}\mathbf{e}, \tag{53}$$

and h_k^{c} is given from Eq. (49) as

$$h_k^{\mathrm{c}} = -\mathbf{e}^T\bar{\mathbf{N}}_2\bar{\hat{\mathbf{g}}}_k. \tag{54}$$

So far, all the unknown constants in the Green's function solutions (Eqs. (36) and (37)) have been determined according to the boundary conditions. It can be verified that the results include those of the electric impermeable model. When $\epsilon_0 = 0$, we have $b^{\mathrm{c}} = 0$ and $\mathbf{H}_\epsilon = \mathbf{0}$. Therefore, Eqs. (52) and (47a), in this case, gives $\mathbf{g}_k = \mathbf{0}$ and $\mathbf{q}_\beta = \mathbf{B}^{-1}\bar{\mathbf{B}}\mathbf{I}_\beta\mathbf{q}$. By substituting them into Eq. (36), the expressions are exactly the results deduced by Lu and Williams (1998).

Generally, the solutions of Eqs. (36) and (37) include infinite series. However, when the cavity is circular, the solutions can be reduced to a closed form. In the case, t in Eq. (31) equals zero. This leads to $y_k = 0$ in Eq. (51), and further $\mathbf{g}_k$ and h_k^{c} all vanish. The infinite series in the solutions therefore disappear.

5. Electric and mechanical fields in material

After the Green's function is obtained, the stresses and the electric displacements in the material, caused by the generalized concentrated forces $\mathbf{p}$, can be determined by Eq. (16), i.e.

$$\begin{Bmatrix}\sigma_{2j}\\ D_2\end{Bmatrix} = \psi_{,1} = \frac{1}{\pi}\,\mathrm{Im}\left\{\mathbf{B}\left[\langle\ln(\varsigma_* - \varsigma_{*0})\rangle_{,1}\mathbf{q} + \sum_{\beta=1}^{4}\langle\ln\Big(\varsigma_*^{-1} - \bar{\varsigma}_{\beta 0}\Big)\rangle_{,1}\mathbf{q}_\beta + \sum_{k=1}^{\infty}\frac{1}{k}\langle\varsigma_*^{-k}\rangle_{,1}\mathbf{g}_k\right]\right\}$$

$$\begin{Bmatrix}\sigma_{1j}\\ D_1\end{Bmatrix} = -\psi_{,2} = -\frac{1}{\pi}\,\mathrm{Im}\left\{\mathbf{B}\left[\langle\ln(\varsigma_* - \varsigma_{*0})\rangle_{,2}\mathbf{q} + \sum_{\beta=1}^{4}\langle\ln\Big(\varsigma_*^{-1} - \bar{\varsigma}_{\beta 0}\Big)\rangle_{,2}\mathbf{q}_\beta + \sum_{k=1}^{\infty}\frac{1}{k}\langle\varsigma_*^{-k}\rangle_{,2}\mathbf{g}_k\right]\right\}, \tag{55}$$

where $\langle\cdot\rangle_{,1} = \partial\langle\cdot\rangle/\partial x_1$ and $\langle\cdot\rangle_{,2} = \partial\langle\cdot\rangle/\partial x_2$.

To determine the hoop stress along the cavity boundary, the generalized hoop stress vector

$$\mathbf{t}_n = -\psi_{,m} \tag{56}$$

has to be obtained at first. Since

$$\frac{\partial}{\partial m} f(\varsigma_\alpha)\bigg|_\Gamma t = -p_\alpha(\theta)\frac{i\sigma f'(\sigma)}{\rho},\ \frac{\partial}{\partial n} f(\varsigma_\alpha)\bigg|_\Gamma = -\frac{i\sigma f'(\sigma)}{\rho}, \tag{57}$$

where

$$p_\alpha(\theta) = \frac{p_\alpha \cos\theta - \sin\theta}{p_\alpha \sin\theta + \cos\theta},\ \rho = (a^2 \sin^2 \vartheta + \cos^2 \vartheta)^{1/2}, \tag{58}$$

we have

$$\frac{\partial}{\partial m}\langle \ln(\varsigma_* - \varsigma_{*0})\rangle\bigg|_\Gamma = \sum_{\alpha=1}^{4} p_\alpha(\theta)\mathbf{I}_\alpha r_\alpha,\ \frac{\partial}{\partial m}\langle \ln\left(\varsigma_*^{-1} - \bar{\varsigma}_{\beta 0}\right)\rangle\bigg|_\Gamma = \sum_{\alpha=1}^{4} p_\alpha(\theta)\mathbf{I}_\alpha \bar{r}_\beta,\ \frac{\partial}{\partial m}\langle \varsigma_*^{-k}\rangle\bigg|_\Gamma$$

$$= \frac{ik\sigma^{-k}}{\rho}\sum_{\alpha=1}^{4} p_\alpha(\theta)\mathbf{I}_\alpha,\ r_\alpha = \frac{-i\sigma}{\rho(\sigma - \varsigma_{\alpha 0})}. \tag{59}$$

Substitution of Eq. (59) into Eq. (56) leads to

$$\mathbf{t}_n = -\frac{1}{\pi}\,\mathrm{Im}\left\{\mathbf{B}\sum_{\alpha=1}^{4} p_\alpha(\theta)\mathbf{I}_\alpha\left[r_\alpha \mathbf{q} + \sum_{\beta=1}^{4}\bar{r}_\beta \mathbf{q}_\beta + \sum_{k=1}^{\infty}\frac{i\sigma^{-k}}{\rho}\mathbf{g}_k\right]\right\}. \tag{60}$$

Therefore, the hoop stresses and the tangential component of the electric displacement along the boundary are given by

$$\sigma_{nn} = n^T \mathbf{I}_0 \mathbf{t}_n,\ \sigma_{nm} = m^T \mathbf{I}_0 \mathbf{t}_n,\ \sigma_{n3} = e_0^T \mathbf{t}_n,\ D_n = \mathbf{e}^T \mathbf{t}_n, \tag{61}$$

where

$$\mathbf{I}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ \mathbf{e}_0 = [0\ 0\ 1\ 0]^T, \tag{62}$$

and $\mathbf{e}$ is defined in Eq. (29).

By letting $b \to 0$ in Eq. (19), the problem discussed before becomes an infinite piezoelectric domain with a slit crack of length $2a$. Therefore, Eqs. (30) and (38b) are reduced to

$$\varsigma_\alpha = \frac{z_\alpha + \sqrt{z_\alpha^2 - a^2}}{a},\ \varsigma_{\alpha 0} = \frac{z_{\alpha 0} + \sqrt{z_{\alpha 0}^2 - a^2}}{a}. \tag{63}$$

On the line ahead of the crack tip, i.e. $x_2 = 0$ and $|x_1| > a$, it gives

$$\varsigma_\alpha = \frac{x_1 + \sqrt{x_1^2 - a^2}}{a} = \chi, \tag{64}$$

which is a real variable. In this case, we have

$$\frac{\partial\ \ln(\varsigma_\beta - \varsigma_{\beta 0})}{\partial x_1} = \frac{1}{\sqrt{x_1^2 - a^2}}\frac{\chi}{\chi - \varsigma_{\beta 0}}, \quad \frac{\partial\ \ln(\varsigma_\alpha^{-1} - \bar{\varsigma}_{\beta 0})}{\partial x_1} = \frac{1}{\sqrt{x_1^2 - a^2}}\frac{1}{\chi\bar{\varsigma}_{\beta 0} - 1}, \quad \frac{\partial \varsigma_\beta^{-k}}{\partial x_1} = -\frac{k\chi^{-k}}{\sqrt{x_1^2 - a^2}}. \tag{65}$$

Therefore, Eq. (55a) on the line $|x_1| > a$ can be written as

$$\begin{Bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \\ D_2 \end{Bmatrix} = \boldsymbol{\psi}_{,1} = \frac{1}{\pi\sqrt{x_1^2 - a^2}}\,\mathrm{Im}\left\{ \sum_{\beta=1}^{4} \frac{\chi}{\chi - \varsigma_{\beta 0}} \mathbf{B}\mathbf{I}_\beta \mathbf{q} + \sum_{\beta=1}^{4} \frac{1}{\chi\bar{\varsigma}_{\beta 0} - 1} \mathbf{B}\mathbf{q}_\beta - \sum_{k=1}^{\infty} \chi^{-k}\mathbf{B}\mathbf{g}_k \right\}. \tag{66}$$

With the definition of stress and electric displacement intensity factors, it gives

$$\mathbf{K} = \begin{Bmatrix} K_{\mathrm{II}} \\ K_{\mathrm{I}} \\ K_{\mathrm{III}} \\ K_{\mathrm{D}} \end{Bmatrix} = \lim_{x_1 \to a} \sqrt{2\pi(x_1 - a)}\,\boldsymbol{\psi}_{,1} = \frac{1}{\sqrt{\pi a}}\,\mathrm{Im}\left\{ \sum_{\beta=1}^{4} \frac{1}{1 - \varsigma_{\beta 0}} \left(\mathbf{B}\mathbf{I}_\beta \mathbf{q} + \bar{\mathbf{B}}\bar{\mathbf{q}}_\beta \right) - \sum_{k=1}^{\infty} \mathbf{B}\mathbf{g}_k \right\}, \tag{67}$$

where K_{I}, K_{II} and K_{III} are the stress intensity factors and K_{D} is the electric displacement intensity factor. Eq. (67) can be simplified by using Eqs. (44b) and (45b):

$$\begin{aligned} \mathbf{K} &= \frac{1}{\sqrt{\pi a}}\,\mathrm{Im}\left\{ \sum_{\beta=1}^{4} \frac{1}{1 - \varsigma_{\beta 0}} \left(2\mathbf{B}\mathbf{I}_\beta \mathbf{q} - b^{\mathrm{c}} q_\beta^{\mathrm{c}} \mathbf{e} \right) - \sum_{k=1}^{\infty} b^{\mathrm{c}} \left(\bar{h}_k^{\mathrm{c}} + h_k^{\mathrm{c}} - \sum_{\beta=1}^{4} \varsigma_{\beta 0}^{-k} q_\beta^{\mathrm{c}} \right) \mathbf{e} \right\} \\ &= \frac{2}{\sqrt{\pi a}}\,\mathrm{Im}\left\{ \mathbf{B} \left\langle \frac{1}{1 - \varsigma_{*0}} \right\rangle \mathbf{A}^T \right\} \mathbf{p} - \frac{2}{\sqrt{\pi a}}\,\mathrm{Im}\left\{ b^{\mathrm{c}} \left(\sum_{\beta=1}^{4} \frac{q_\beta^{\mathrm{c}}}{1 - \varsigma_{\beta 0}} + \sum_{k=1}^{\infty} h_k^{\mathrm{c}} \right) \right\} \mathbf{e}, \end{aligned} \tag{68}$$

in which Eq. (38a) and $t = 1$ when $b = 0$ as well as the series representation

$$\sum_{k=1}^{\infty} x^{-k} = \frac{1}{x - 1}, \quad |x| > 1, \tag{69}$$

have also been used. It can be seen from Eq. (68) that the first term on the right side of the equation is just the result obtained by Lu and Williams (1998) for the conventional electric impermeable boundary conditions, and the rest of the terms only contribute to the electric displacement intensity factor K_{D}. It means that the stress intensity factors do not rely on the use of the electric boundary conditions. This conclusion is consistent with the case reported for a piezoelectric body with a crack subject to some simple mechanical and electrical loading (Zhang et al., 1998).

Eq. (68) can be further simplified. From Eqs. (54), (51) and (47b), we have

$$\mathrm{Im}\sum_{k=1}^{\infty} b^{c} h_k^{c} = -\mathrm{Im}\left\{ \mathbf{e}^T \bar{\mathbf{N}}_2 \left(\bar{\mathbf{N}}_1 - \mathbf{N}_2 \mathbf{N}_1^{-1} \bar{\mathbf{N}}_2 \right)^{-1} \sum_{k=1}^{\infty} \sum_{\beta=1}^{4} b^{c} \left(\bar{\varsigma}_{\beta 0}^{-k} \bar{q}_\beta^{c} + \varsigma_{\beta 0}^{-k} q_\beta^{c} \mathbf{N}_2 \mathbf{N}_1^{-1} \right) \mathbf{e} \right\}$$

$$= \mathrm{Im}\left\{ b^{c} G \sum_{\beta=1}^{4} \frac{1}{1-\varsigma_{\beta 0}} q_\beta^{c} \right\}, \tag{70}$$

where

$$G = \mathbf{e}^T \left\{ \mathbf{N}_2 \left(\mathbf{N}_1 - \bar{\mathbf{N}}_2 \bar{\mathbf{N}}_1^{-1} \mathbf{N}_2 \right)^{-1} + \bar{\mathbf{N}}_2 \left(\bar{\mathbf{N}}_1 - \mathbf{N}_2 \mathbf{N}_1^{-1} \bar{\mathbf{N}}_2 \right)^{-1} \mathbf{N}_2 \mathbf{N}_1^{-1} \right\} \mathbf{e}. \tag{71}$$

Substitution of Eq. (70) into Eq. (68) yields

$$\mathbf{K} = \frac{2}{\sqrt{\pi a}} \mathrm{Im}\left\{ \mathbf{B} \langle \frac{1}{1-\varsigma_{*0}} \rangle \mathbf{A}^T \right\} \mathbf{p} - \frac{2}{\sqrt{\pi a}} \mathrm{Im}\left\{ b^{c}(1+G) \sum_{\beta=1}^{4} \frac{q_\beta^{c}}{1-\varsigma_{\beta 0}} \right\} \mathbf{e}$$

$$= \frac{2}{\sqrt{\pi a}} \mathrm{Im}\left\{ \left[\mathbf{B} - b^{c}(1+G) \mathbf{I}_4 \left(\mathbf{A} - \bar{\mathbf{A}} \bar{\mathbf{N}}_1^{-1} \mathbf{N}_2 \right) \right] \langle \frac{1}{1-\varsigma_{*0}} \rangle \mathbf{A}^T \right\} \mathbf{p}. \tag{72}$$

It can be seen that when $\epsilon_0 = 0$ or $b^{c} = 0$, the intensity factors reduce to the conventional form (Lu and Williams, 1998).

6. Electric fields in cavity

After the constants q_β^{c} and h_k^{c} have been known from Eqs. (47) and (54), the electric potential inside the cavity can be obtained by Eq. (37a), and the components of the electric field in Ω_c can be determined as

$$E^{c} = E_1^{c} - iE_2^{c} = -\phi_{,1}^{c} + i\phi_{,2}^{c} = i\frac{1}{\pi} \left\{ \sum_{\beta=1}^{4} \frac{1}{z-\hat{z}_\beta} q_\beta^{c} + \sum_{k=1}^{\infty} \frac{\varsigma^k - t^k \varsigma^{-k}}{\sqrt{z^2 - a^2 + b^2}} h_k^{c} \right\}, \tag{73}$$

where z, ς and t are defined by Eq. (31).

Now we begin to determine the normal component of the electric displacements, and verify the continuity condition of the component on the curve Γ. According to the derivative relation Eq. (57b), we have

$$\left. \frac{\partial \ln(z-\hat{z}_\beta)}{\partial n} \right|_\Gamma = \frac{\partial}{\partial n} \ln\left[\frac{1}{2}(a+b)(\varsigma - \varsigma_{\beta 0})(1 - \hat{\tau}_\beta \varsigma^{-1}) \right] \Bigg|_\Gamma = -i\frac{1}{\rho} \left(\frac{\sigma}{\sigma - \varsigma_{\beta 0}} + \frac{1}{\sigma \hat{\tau}_\beta^{-1} - 1} \right),$$

$$\left. \frac{\partial}{\partial n} f_k(\varsigma) \right|_\Gamma = \left. \frac{\partial}{\partial n} (\varsigma^k + t^k \varsigma^{-k}) \right|_\Gamma = -i\frac{k}{\rho}(\sigma^k - t^k \sigma^{-k}), \tag{74}$$

and

$$\frac{\partial}{\partial n}\langle \ln(\varsigma_* - \varsigma_{*0})\rangle\Big|_\Gamma = \sum_{\beta=1}^{4} r_\beta \mathbf{I}_\beta,\ \frac{\partial}{\partial n}\langle \ln(\varsigma_*^{-1} - \bar{\varsigma}_{\beta 0})\rangle\Big|_\Gamma = \bar{r}_\beta \mathbf{I},$$

$$\frac{\partial}{\partial n}\langle \varsigma_*^{-k}\rangle\Big|_\Gamma = \frac{ik\sigma^{-k}}{\rho}\mathbf{I},\ r_\beta = \frac{-i\sigma}{\rho(\sigma - \varsigma_{\beta 0})}. \tag{75}$$

Substitution of Eqs. (74) and (37b) into Eq. (27) leads to

$$D_m^c = \psi_{,n}^c = -\frac{1}{\pi}\,\mathrm{Im}\left\{\sum_{\beta=1}^{4}\frac{ib^c}{\rho}\left(\frac{\sigma}{\sigma - \varsigma_{\beta 0}} + \frac{1}{\sigma\hat{\tau}_\beta^{-1} - 1}\right)q_\beta^c + \sum_{k=1}^{\infty}\frac{ib^c}{\rho}(\sigma^k - t^k\sigma^{-k})h_k^c\right\}, \tag{76}$$

which is the normal component of the electric displacements on the curve Γ, obtained by the expression for the electric potential inside the cavity. On the other hand, by substituting Eqs. (75) and (36b) into Eq. (21), one has

$$\mathbf{t}_m = \psi_{,n} = \frac{1}{\pi}\,\mathrm{Im}\left\{\sum_{\beta=1}^{4} r_\beta \mathbf{B}\mathbf{I}_\beta \mathbf{q} + \sum_{\beta=1}^{4}\bar{r}_\beta \mathbf{B}\mathbf{q}_\beta + \sum_{k=1}^{\infty}\frac{i\sigma^{-k}}{\rho}\mathbf{B}\mathbf{g}_k\right\}$$

$$= \frac{1}{\pi}\,\mathrm{Im}\left\{-\sum_{\beta=1}^{4}\bar{r}_\beta \bar{b}^c \bar{q}_\beta^c \mathbf{e} + \sum_{k=1}^{\infty}\frac{ib^c\sigma^{-k}}{\rho}\left(\bar{h}_k^c + t^k h_k^c - \sum_{\beta=1}^{4}\hat{\tau}_\beta^k q_\beta^c\right)\mathbf{e}\right\}, \tag{77}$$

in which Eqs. (44b) and (45b) have been used. The first three components of $\mathbf{t}_m$ in Eq. (77) are seen to be zero. It is consistent with the prescribed boundary conditions (28a). Eq. (77) can be further simplified as

$$\mathbf{t}_m = \frac{1}{\pi}\,\mathrm{Im}\left\{\sum_{\beta=1}^{4} r_\beta b^c q_\beta^c \mathbf{e} + \sum_{k=1}^{\infty}\frac{ib^c}{\rho}(-\sigma^k + t^k\sigma^{-k})h_k^c\mathbf{e} - \sum_{\beta=1}^{4}\frac{ib^c}{\rho}\frac{1}{\sigma\hat{\tau}_\beta^{-1} - 1}q_\beta^c\mathbf{e}\right\} = D_m^c\mathbf{e}. \tag{78}$$

It shows that $D_m = D_m^c$. This from another side confirms the correctness of the results obtained in the paper. It can also be seen that $D_m^c = 0$ when $\epsilon_0 = 0$.

7. Concluding remarks

The Green's functions of an infinite two-dimensional piezoelectric material containing an elliptical cavity are re-investigated by introducing exact electrical boundary conditions on the hole boundary, and corresponding modified solutions are obtained in general form. By setting ϵ_0, the dielectric constant in the cavity, to be zero, the modified Green's functions are returned to the conventional ones (Lu and Williams, 1998). When the cavity is reduced to a slit crack, the results show that the values of the stress intensity factors on the crack tip by using the present solutions are same as those obtained based on the electric impermeable condition. It means that the stress intensity factors do not rely on the use of the electric boundary conditions. It is also verified that the normal components of the electrical displacements obtained, respectively, by the expressions for the electrical potential inside the cavity and in the material are indeed the same on the curve boundary. This confirms the correctness of the results in the paper.

Since the general electrical boundary conditions are introduced, the modified Green's functions are more reasonable physically. The results are of importance for both analytical and numerical analyses for piezoelectric materials with defects. For example, with the solutions, one can investigate the influence of the electrical fields inside the cavity to the mechanical and electrical properties of the considered problems. The solutions can also be used as kernels of boundary integral equations in BEM analyses. Since the Green functions have satisfied the boundary conditions of the hole or crack, the boundary integrals along these surfaces can be avoided (Mukherjee, 1982), which could save a lot of numerical computing efforts in the BEM approach.

Acknowledgements

The authors are grateful for the support provided by the National Nature Science Foundation of China and the School of Mechanical and Production Engineering, Nanyang Technological University of Singapore.

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